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Wiener–Hopf factorization for a group of exponentials of nilpotent matrices

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Abstract

A complete study of the generalized factorization for a group of 2×2 matrix functions of the form $G = I + \gamma N$, where $\gamma \in \mathscr{C}(\dot{\mathbb{R}})$, I denotes the 2×2 identity matrix and N represents a rational nilpotent matrix function, is presented. A closely related class involving the same matrix N is also studied. The canonical and non-canonical factorizations are considered and explicit formulas are obtained for the partial indices and the factors in such factorizations. It is shown in particular that only one of the columns in the factors needs to be determined, as a solution to a homogeneous linear Riemann–Hilbert problem, the other column being expressed in terms of the first. Necessary and sufficient conditions for existence of a canonical factorization within the same class are established, as well as explicit formulas for the factors in this case. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

The concept of generalized factorization relative to $L_p(\mathbb{R})$ $(p > 1)$, which is included in what is commonly known as Wiener–Hopf factorization, plays a fundamen-

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tal role in many problems, namely in the theory of Wiener–Hopf equations. However, there is no general method to obtain such a factorization for a matrix function $G \in (L_\infty(\mathbb{R}))^{n\times n}$, unless it is rational or $n = 1$. Only certain classes of matrix functions have been studied, such as those reducible to triangular form [17] or the so-called 2×2 K-form matrices [15],

$$G = \alpha I + \beta R, \tag{1.1}$$

where $\alpha, \beta \in \mathscr{C}(\dot{\mathbb{R}})$, I denotes the identity matrix and R is rational matrix with $\operatorname{tr} R = 0$.

A class of matrices of this form which has received considerable attention in the literature is the Daniele–Khrapkov class, for which $\det R \neq 0$ [5,6,8,11–13,15]. R is often assumed to be of the following simple form:

$$R = \begin{bmatrix} 0 & 1 \\ q & 0 \end{bmatrix}, \tag{1.2}$$

q being a non-zero rational function without poles on $\mathbb{R}$.

Another class of K-form matrices is the so-called *class* $\mathscr{N}$ [3], where R is nilpotent:

$$G = \alpha I + \beta N \quad \text{with } N = \begin{bmatrix} 1 & q \\ -q^{-1} & -1 \end{bmatrix}. \tag{1.3}$$

In this paper, we consider the generalized factorization of such matrices in connection with a group of matrices of the form $I + \gamma N$, $\gamma \in C(\dot{\mathbb{R}})$, for which necessary and sufficient conditions for existence of canonical generalized factorization and canonical factorization with factors in the same group, as well as explicit formulas for the factors, are determined. Non-canonical factorization is also studied.

This class of matrix functions presents some interesting characteristics which distinguish it from the Daniele–Khrapkov class, in spite of the formal similarity. In particular, the conditions for existence of a canonical factorization are quite different. It turns out that, even for the simple case where $\operatorname{ind} G = 0$ and q is a quotient of two first-degree polynomials without common zeros, G may admit a non-canonical factorization. This shows some similarity with other classes of matrices, such as the more complicated case of Daniele–Khrapkov matrices where q is a quotient of polynomials with degree greater than 1.

A particular case of matrices belonging to the class considered in this paper has been studied in [3], where conditions for existence of a canonical factorization and explicit formulas for the factors are obtained. The factorization method presented here represents a development of the one used in [3]. This development points to greater simplicity and generality, since factorization within the class and non-canonical factorization are also considered, which has not been done before.

An interesting feature is that a meromorphic factorization [1,2], allowing the factors to present poles in the corresponding half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$, arises naturally in this case. A linear algebraic procedure to turn it into a generalized factorization is then applied to obtain a canonical factorization.

It is shown that one of the columns in the factors $G_{\pm}$ can always be expressed in terms of the other one. Thus, contrary to what has been done in a previous paper concerning the canonical factorization of certain matrices in the same class $\mathscr{N}$ [3], only one of the columns in the factors $G_{\pm}$ is determined from a solution to the linear equation

$$G\,\phi^{+} = \tilde{r}\,\phi^{-}, \quad \phi^{\pm} \in \left(L_2^{\pm}(\mathbb{R})\right)^2, \tag{1.4}$$

where $\tilde{r}(\xi) = (\xi - b)/(\xi + b)$, $b \in \mathbb{C}^{+}$, the other column being expressed in terms of the first. Another simplifying change is that this solution to (1.4) is obtained directly from the non-trivial solutions of the equation $G\,\phi^{+} = \phi^{-}$, used to determine the partial indices.

The authors are convinced that the method used here can be applied to obtain the generalized factorization of new classes of matrix functions or to simplify the study of others.

2. Preliminaries

We start by defining some classes of functions which will be used later.

Let P^{+} and P^{-} be the complementary projections on $L_2(\mathbb{R})$ given by

$$P^{\pm} = \tfrac{1}{2}\,(I \pm S_{\mathbb{R}}), \tag{2.1}$$

where I denotes the identity operator and $S_{\mathbb{R}}$ denotes the singular integral operator in $L_2(\mathbb{R})$ defined by

$$S_{\mathbb{R}}\, f(\xi) = \frac{1}{\mathrm{i}\pi} \int_{\mathbb{R}} \frac{f(t)}{t - \xi}\,\mathrm{d}t, \quad \xi \in \mathbb{R}, \tag{2.2}$$

the integral being understood in the sense of Cauchy's principal value. Let moreover $\mathbb{C}^{\pm} = \{z \in \mathbb{C}\colon \mathrm{Im}(z) \in \mathbb{R}^{\pm}\}$.

We denote by L_2^{+} and L_2^{-} the images of P^{+} and P^{-}, respectively. A function f in L_2^{+} (resp. L_2^{-}) can be identified with a function in the Hardy space $H^2(\mathbb{C}^{+})$ (resp. $H^2(\mathbb{C}^{-})$), which we denote by the same symbol f. By L_∞^{+} (resp. L_∞^{-}) we denote the space of all essentially bounded functions $f \in L_\infty(\mathbb{R})$ which admit a bounded analytic extension (in the sense of Hardy spaces) to the half-plane $\mathbb{C}^{+}$ (resp. $\mathbb{C}^{-}$).

$\mathscr{C}(\dot{\mathbb{R}})$ represents the algebra of all functions which are continuous in $\mathbb{R}$ and possess equal limits at $\pm\infty$ and by $\mathscr{R}(\dot{\mathbb{R}})$ the space of all rational functions in $\mathscr{C}(\dot{\mathbb{R}})$.

$\mathscr{H}(\mathbb{C}^{\pm})$ represents the class of all complex-valued functions defined on $\mathbb{R}$, which admit an analytic extension into $\mathbb{C}^{\pm}$.

If $\mathscr{A}$ is an algebra, we denote by $\mathscr{G}(\mathscr{A})$ the group of invertible elements in $\mathscr{A}$ and by $X^{n\times n}$ the space of $n \times n$ matrix functions with entries from a linear space X.

Now let $G \in \mathscr{G}\left(L_\infty(\mathbb{R})^{n\times n}\right)$. By a *generalized factorization* of G (relative to $L_2(\mathbb{R})$) we mean a factorization of the form

$$G = G_{-}\,D\,G_{+} \tag{2.3}$$

with $r_- \, G_-^{\pm 1} \in (L_2^-)^{2\times 2}$, $r_+ \, G_+^{\pm 1} \in (L_2^+)^{n\times n}$ and $D = \operatorname{diag}\big(r^{k_j}\big)_{j=1,2,\dots,n}$, where $k_j \in \mathbb{Z}$ $(j = 1, 2, \dots, n)$,

$$r(\xi) = \frac{\xi - \mathrm{i}}{\xi + \mathrm{i}}, \quad r_\pm(\xi) = \frac{1}{\xi \pm \mathrm{i}} \qquad (\xi \in \mathbb{R}), \tag{2.4}$$

and such that $G_+^{-1} \, P^+ \, G_-^{-1} \, I$ is an operator defined on a dense subset of $(L_2(\mathbb{R}))^n$ and possessing a bounded extension to $(L_2(\mathbb{R}))^n$. In (2.4), we can replace the complex number i by any complex number $b \in \mathbb{C}^+$ without changing the main results.

We say that G is 2-*regular* if it admits a generalized factorization relative to $L_2(\mathbb{R})$.

We assume here that the integers k_j $(j = 1, 2, \dots, n)$, called *partial indices* of G, are such that $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$. The *total index* of G is given by $\operatorname{ind} G = \sum_{j=1}^n k_j$. In particular, for $n = 2$, $\operatorname{ind} G = 0$ implies that the partial indices k_1, k_2 are symmetric $(k_1 = -k_2)$. The factorization is said to be *canonical* if $k_1 = k_2 = 0$ and *bounded* if $G_-^{\pm 1} \in (L_\infty^-)^{n\times n}$, $G_+^{\pm 1} \in (L_\infty^+)^{n\times n}$.

If $G \in \mathscr{G}(\mathscr{C}(\dot{\mathbb{R}}))^{n\times n}$, then it admits a factorization of the form (2.3) and, in that case, det G also admits a generalized factorization

$$\det G = (\det G_-) \; r^k \; (\det G_+),$$

where $k = \operatorname{ind} G$ and this value coincides with the index of $\det(G)$ (as a continuous function in $\dot{\mathbb{R}}$) [7,14].

The existence and the characteristics of a generalized factorization for G (relative to $L_2(\mathbb{R})$) determine the Fredholm properties of the operator

$$T_G : \left(L_2^+(\mathbb{R})\right)^n \to \left(L_2^+(\mathbb{R})\right)^n, \quad T_G = P^+ \, G \, I_+,$$

where I_+ denotes the identity operator in $\left(L_2^+(\mathbb{R})\right)^n$. T_G is a Fredholm operator iff G admits a generalized factorization relative to $L_2(\mathbb{R})$ and, in this case, $\dim\ker T_G = -\sum_{k_i<0} k_i$, $\dim\operatorname{coker} T_G = \sum_{k_i>0} k_i$. In particular, T_G is invertible iff G admits a canonical factorization, its inverse being (cf. [7,14]):

$$G_+^{-1} \, P^+ \, G_-^{-1} \, I_+ : (L_2^+(\mathbb{R}))^n \to (L_2^+(\mathbb{R}))^n.$$

Another concept of matrix factorization, which allows the factors to have algebraic behaviour at ∞ and poles in the corresponding half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ is the following. We say that $G \in \mathscr{G}\big(L_\infty(\mathbb{R})^{2\times 2}\big)$ admits a *meromorphic factorization* (relative to $L_2(\mathbb{R})$) if

$$G = M_- \; R \; M_+$$

with $R \in \mathscr{G}\big(\mathscr{R}(\mathbb{R})^{2\times 2}\big)$ and $M_\pm$ are such that there are polynomials $p_\pm, q_\pm$, without zeros in $\overline{\mathbb{C}^\pm}$, respectively, such that

$$r_+ \operatorname{diag}(r_+^{-m_1}, \, r_+^{-m_2}) \; (r_+^{\deg q_-} \; q_- \; M_+) \in (L_2^+)^{2\times 2},$$
$$r_+ \, (r_+^{\deg p_-} \; p_- \; M_+^{-1}) \operatorname{diag}(r_+^{m_1}, \, r_+^{m_2}) \in (L_2^+)^{2\times 2},$$

$$r_- \, (r_-^{\deg q_+} \; q_+ \, M_-) \; \mathrm{diag}\,(r_-^{m_1}, \; r_-^{m_2}) \in (L_2^-)^{2\times 2},$$
$$r_- \; \mathrm{diag}\,(r_-^{-m_1}, \; r_-^{-m_2}) \; (r_-^{\deg p_+} \; p_+ \, M_-^{-1}) \in (L_2^-)^{2\times 2}$$

with $m_1, m_2 \in \mathbb{Z}$.

The concept of meromorphic factorization was first introduced in [1] for 2×2 matrix functions with entries from $C^{\mu}(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$ denotes the two-point compactification of $\mathbb{R}$. It was further developed in [2] for $n \times n$ matrix functions in $\mathscr{G}\left(L^{\infty}(\mathbb{R})^{n\times n}\right)$. We use it here in a slightly modified form for convenience.

It was shown in [2] that from a meromorphic factorization of G, such as defined above, we can obtain a generalized factorization for G, if this matrix function is 2-regular. A procedure to do that explicitly (which is based on the simple idea that if the determinant of an $n \times n$ matrix is zero, its columns are linearly dependent) was presented in [2] and will be used later in this paper.

3. The class $\mathscr{N}$ and the group $\mathscr{N}_0$

Let G be a 2×2 regular matrix function of the form

$$G = \alpha \, I + \beta \, N, \tag{3.1}$$

where $\alpha, \beta \in \mathscr{C}(\dot{\mathbb{R}})$ and N is a non-zero nilpotent matrix, whose entries are rational functions without poles on $\mathbb{R}$.

We will assume that N is of the form

$$N = \begin{bmatrix} 1 & q \\ -q^{-1} & -1 \end{bmatrix}, \quad q \in \mathscr{R}(\dot{\mathbb{R}}). \tag{3.2}$$

In fact, it is not difficult to see that if N is a nilpotent matrix

$$N = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}$$

with $q_j \in \mathscr{R}(\dot{\mathbb{R}})$, the matrix function G (given by (3.1)) is non-triangular only if

$$q_1 + q_4 = 0, \quad q_1 \neq 0, \quad q_1^2 + q_2 \, q_3 = 0.$$

This means that

$$\begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} = q_1 \begin{bmatrix} 1 & q \\ -q^{-1} & -1 \end{bmatrix} \quad \text{for } q = \frac{q_2}{q_1}$$

and, if q_1 does not vanish on $\mathbb{R}$, we have $q \in \mathscr{R}(\dot{\mathbb{R}})$.

We will consider in this paper only the case where q is a quotient of two first-degree polynomials with different non-real zeros a and b ($b \in \mathbb{C}^+$, for instance),

$$q(\xi) = \frac{\xi + a}{\xi + b} \quad (\xi \in \mathbb{R}) \tag{3.3}$$

and that α admits a bounded factorization.

The class of all such matrices G will be denoted by $\mathscr{N}$.

It is enough, on the other hand, to study the generalized factorization of G in the case where $\operatorname{ind} G = 0$. In fact, since $\det G = \alpha^2$, $\operatorname{ind} G$ must be an even integer; let $\operatorname{ind} G = 2k$, $k \in \mathbb{Z}$. Defining

$$\tilde{G} = r^{-k} G$$

we see that $\tilde{G}$ has the same form as G, with α and β replaced by $r^{-k}\alpha$ and $r^{-k}\beta$, respectively, and $\operatorname{ind} \tilde{G} = 0$. It is obvious that a generalized factorization for G can be immediately obtained from a factorization of the same type for $\tilde{G}$.

Thus let G be of the form

$$G = \begin{bmatrix} \alpha + \beta & \beta q \\ -\beta q^{-1} & \alpha - \beta \end{bmatrix}, \tag{3.4}$$

where q is defined by (3.3), $\alpha, \beta \in \mathscr{C}(\dot{\mathbb{R}})$ and α admits a canonical bounded factorization

$$\alpha = \alpha_- \alpha_+. \tag{3.5}$$

With these assumptions, it is clear that G admits a generalized factorization (since it is invertible in $C(\dot{\mathbb{R}})^{2\times 2}$) $G = G_- D G_+$. It is also clear that $\tilde{G} = \alpha^{-1} G = I + \gamma N$, with $\gamma = \beta/\alpha$, admits a factorization of the same type, $\tilde{G} = \tilde{G}_- D \tilde{G}_+$, where $\tilde{G}_- = \alpha_-^{-1} G_-$, $\tilde{G}_+ = \alpha_+^{-1} G_+$.

Thus we see that studying the generalized factorization for matrix functions of class $\mathscr{N}$ is equivalent to studying the generalized factorization for matrices of the form

$$G = I + \gamma N, \quad \gamma \in C(\dot{\mathbb{R}}). \tag{3.6}$$

We remark that (3.6) is equivalent to

$$G = \exp(\gamma N), \quad \gamma \in C(\dot{\mathbb{R}}), \tag{3.7}$$

and the class of all matrix functions of this form is a multiplicative group, which we denote by $\mathscr{N}_0$. One-parameter groups of the same type have appeared in connection with integrable systems [10,16], which constitutes another motivation for studying matrix functions of this form.

In the following sections we consider the Wiener–Hopf factorization of matrices in this group. We start by considering the partial indices in a generalized factorization for G, which includes determining necessary and sufficient conditions for existence of a canonical factorization.

4. Partial indices

Let us consider the homogeneous Riemann–Hilbert problem

$$G\phi^+ = \phi^-, \quad \phi^\pm = \left(\phi_1^\pm, \phi_2^\pm\right) \in (L_2^\pm)^2 \tag{4.1}$$

for $G = I + \gamma N \in \mathcal{N}_0$. Since $\operatorname{ind} G = 0$, G admits a canonical factorization iff (4.1) has no solutions different from zero. We start by establishing necessary and sufficient conditions for this to happen.

Theorem 4.1. *Let $G \in \mathcal{N}_0$ be a matrix function of the form* (3.6). *G admits a canonical factorization iff*

$$\left[P^-\left(\frac{\gamma}{\xi + a}\right)\right]_{\xi=-b} \neq \frac{1}{b-a}. \tag{4.2}$$

Proof. Eq. (4.1) is equivalent to

$$\begin{aligned} &(1+\gamma)\,\phi_1^+ + \gamma\, q\, \phi_2^+ = \phi_1^-, \\ &-\gamma\,\phi_1^+ + (1-\gamma) q\, \phi_2^+ = q\, \phi_2^- \end{aligned} \tag{4.3}$$

and, by adding these two equations, we obtain

$$\phi_1^+ + q\,\phi_2^+ = \phi_1^- + q\,\phi_2^-. \tag{4.4}$$

Due to the pole $-b \in \mathbb{C}^-$ on the right-hand side, it follows that both sides of (4.4) must be of the form $K/(\xi + b)$, where $K \in \mathbb{C}$, and thus we have

$$\phi_1^+ + q\,\phi_2^+ = \frac{K}{\xi + b}, \tag{4.5}$$

$$\phi_1^- + q\,\phi_2^- = \frac{K}{\xi + b}. \tag{4.6}$$

If $K = 0$, it follows from (4.5) and (4.3) that $\phi^+ = \phi^- = 0$. Therefore, the existence of non-trivial solutions to (4.1) implies that (4.5) and (4.6) are satisfied with $K \neq 0$.

Let us assume that there are non-zero solutions (ϕ^+, ϕ^-) to (4.1). Then, from the second equation in (4.3) and (4.5) it follows that

$$\phi_2^- = \phi_2^+ - \gamma\,\frac{K}{\xi + a} \tag{4.7}$$

and thus

$$\phi_2^- = -K\,P^-\left(\frac{\gamma}{\xi + a}\right). \tag{4.8}$$

Taking now (4.6) into account we see that we must have

$$\phi_1^- = \frac{K}{\xi + b}\left[1 + (\xi + a)P^-\left(\frac{\gamma}{\xi + a}\right)\right], \tag{4.9}$$

and therefore a necessary condition for existence of non-trivial solutions to (4.1) is

$$\left[1 + (\xi + a)P^-\left(\frac{\gamma}{\xi + a}\right)\right]_{(-b)} = 0,$$

which is equivalent to

$$\left[P^{-}\left(\frac{\gamma}{\xi+a}\right)\right]_{(-b)}=\frac{1}{b-a}.$$

Conversely, if this condition is satisfied, we have from (4.7)

$$\phi_2^{+}=K\,P^{+}\left(\frac{\gamma}{\xi+a}\right) \tag{4.10}$$

and, using (4.5),

$$\phi_1^{+}=\frac{K}{\xi+b}\left[1-(\xi+a)\,P^{+}\left(\frac{\gamma}{\xi+a}\right)\right], \tag{4.11}$$

where $K\neq 0$. Defining $\phi^{\pm}=(\phi_1^{\pm},\ \phi_2^{\pm})$, where $\phi_1^{\pm},\ \phi_2^{\pm}$ are given by (4.8)–(4.11), we see that $(\phi^{+},\ \phi^{-})$ is a non-trivial solution to $G\,\phi^{+}=\phi^{-}$. □

Corollary 4.2. *If condition* (4.2) *is not satisfed*, *the solutions to* (4.1) *are given by*

$$\phi_1^{+}=\frac{K}{\xi+b}\left(1-(\xi+a)\,P^{+}\left(\frac{\gamma}{\xi+a}\right)\right), \tag{4.12}$$

$$\phi_2^{+}=K\,P^{+}\left(\frac{\gamma}{\xi+a}\right), \tag{4.13}$$

$$\phi_1^{-}=\frac{K}{\xi+b}\left(1+(\xi+a)\,P^{-}\left(\frac{\gamma}{\xi+a}\right)\right), \tag{4.14}$$

$$\phi_2^{-}=-K\,P^{-}\left(\frac{\gamma}{\xi+a}\right), \tag{4.15}$$

where $K\in\mathbb{C}$.

Corollary 4.3. *The partial indices in a generalized factorization for* G *are* $k_1=k_2=0$ *iff condition* (4.2) *is satisfied. In this case the operator* $T_G:\left(L_2^{+}(\mathbb{R})\right)^2\to\left(L_2^{+}(\mathbb{R})\right)^2,\ T_G=P^{+}\,G\,I_{+}$, *is invertible. Otherwise we have*

$$\dim\ker T_G=1$$

and the partial indices in a generalized factorization for G *are* $k_1=-1,\ k_2=1$.

Taking into account what was said in Section 3, for the more general case, where $\operatorname{ind}\alpha=k$, we have the following:

Corollary 4.4. *Let* $G\in\mathcal{N}$ *be a matrix of the form* (3.1) *and let* α *admit a bounded factorization* $\alpha=\alpha_{-}\,r^{k}\,\alpha_{+}$. *Then* $\operatorname{ind}G=2k$ *and*

$$k_1=k_2=k \quad \textit{iff} \quad \left[P^{-}\left(\frac{\gamma}{\xi+a}\right)\right]_{(-b)}\neq\frac{1}{b-a},$$

$$k_1 = k-1,\;\; k_2 = k+1 \quad \textit{iff} \quad \left[P^-\left(\frac{\gamma}{\xi+a}\right)\right]_{(-b)} = \frac{1}{b-a},$$

where $\gamma = \beta/\alpha$.

In the following two sections we turn to the determination of a canonical factorization for $G \in \mathscr{N}_0$, if (4.2) is satisfied, and a non-canonical factorization, if condition (4.2) is not fulfilled. However it is important to remark here that, in both cases, we will use the results of Corollary 4.2, interpreted as in the following remark.

Remark 4.5. The proof of Theorem 4.1 shows that Eq. (4.1) admits non-trivial solutions iff

$$\left[P^-\left(\frac{\gamma}{\xi+a}\right)\right]_{(-b)} = \frac{1}{b-a}.$$

This condition arises from equality (4.9) and from the fact that ϕ_1^- must belong to L_2^-. However, if we admit for the equation $G\,\phi^+ = \phi^-$ solutions in $\left(L_2^\pm + \mathscr{R}(\mathbb{R})\right)^2$, we see that $\phi^\pm = \left(\phi_1^\pm,\ \phi_2^\pm\right)$, with $\phi_1^\pm,\ \phi_2^\pm$ defined by (4.8)–(4.11), satisfy this equation even if condition (4.2) is fulfilled.

In fact, in this case, $\phi^\pm$ defined by (4.12)–(4.15) are the solutions to the following problem: "Determine $\phi^\pm$ such that $G\,\phi^+ = \phi^-$ and $\phi_1^+,\ \phi_2^+ \in L_2^+,\ \phi_2^- \in L_2^-,\ \tilde{r}^{-1}\,\phi_1^- \in L_2^-$", where $\tilde{r}(\xi) = (\xi-b)/(\xi+b)$.

This interpretation of the previous results will later allow us to obtain one of the columns in G_+^{-1} and G_-, when G admits a canonical factorization $G = G_-\,G_+$ (which happens when condition (4.2) is satisfied) from the results of Corollary 4.2 (which were obtained assuming that condition (4.2) is not satisfied).

We now state these conclusions as a separate result.

Theorem 4.6. *The solutions to* $G\,\phi^+ = \psi$, *where* $\phi^+ = \left(\phi_1^+,\ \phi_2^+\right) \in \left(L_2^+\right)^2$ *and* $\psi = (\psi_1,\ \psi_2)$ *with* $\tilde{r}^{-1}\,\psi_1 \in L_2^-,\ \psi_2 \in L_2^-$ *are of the form*:

$$\phi_1^+ = \frac{K}{\xi+b}\left(1-(\xi+a)P^+\left(\frac{\gamma}{\xi+a}\right)\right), \tag{4.16}$$

$$\phi_2^+ = K\,P^+\left(\frac{\gamma}{\xi+a}\right), \tag{4.17}$$

$$\psi_1 = \frac{K}{\xi+b}\left(1+(\xi+a)P^-\left(\frac{\gamma}{\xi+a}\right)\right), \tag{4.18}$$

$$\psi_2 = -K\,P^-\left(\frac{\gamma}{\xi+a}\right), \tag{4.19}$$

where $K \in \mathbb{C}$.

Proof. It is enough to follow the proof of Theorem 4.1, taking the conditions on ψ_1 and ψ_2 into account. □

5. Canonical factorization

Let $G \in \mathscr{N}_0$ be a matrix function of the form (3.6) such that (4.2) is satisfied. In order to determine the factors in a canonical factorization $G = G_- G_+$, we now study the Riemann–Hilbert problem

$$G\,\phi^+ = \tilde{r}\,\phi^-, \quad \phi^\pm \in \left(L_2^\pm\right)^2, \tag{5.1}$$

where

$$\tilde{r}(\xi) = \frac{\xi - b}{\xi + b} \qquad (b \in \mathbb{C}^+).$$

In this case, Eq. (5.1) admits two linearly independent solutions, from which we obtain a canonical factorization for G. In fact, we can determine the two columns in G_+^{-1} and G_- separately by solving the homogeneous equation (5.1) with convenient normalizing conditions (chosen in order to ensure the linear independence of the two columns, while keeping the problem as simple as possible). If (ϕ^+, ϕ^-) is a non-trivial solution to $G\,\phi^+ = \tilde{r}\,\phi^-$, then $(\xi + b)\phi^+$ and $(\xi - b)\phi^-$ can be taken as being one of the columns in G_+^{-1} and G_-, respectively [3].

We will show, however, that in this case we only need to determine one of the columns by this procedure, since the second column can be expressed in terms of the first.

We start by establishing some properties of a solution to (5.1) satisfying certain (convenient) normalizing conditions.

Theorem 5.1. *Let* $\left(\phi^+, \phi^-\right)$ *be a solution to* (5.1) *such that*

$$\phi_1^-(-b) \neq 0, \quad \phi_2^-(-b) = 0. \tag{5.2}$$

The following equalities hold:

$$(\xi + b)\,(\phi_1^+ + q\,\phi_2^+) = K, \quad (\xi - b)\,(\phi_1^- + q\,\phi_2^-) = K, \tag{5.3}$$

where $K \in \mathbb{C}\backslash\{0\}$.

Proof. The equation $G\,\phi^+ = \tilde{r}\,\phi^-$ can be written in the form

$$\begin{aligned} &(1+\gamma)\,\phi_1^+ + \gamma\,q\,\phi_2^+ = \tilde{r}\,\phi_1^-, \\ &-\gamma\,\phi_1^+ + (1-\gamma)q\,\phi_2^+ = q\,\tilde{r}\,\phi_2^- \end{aligned} \tag{5.4}$$

and, as in the proof of Theorem 5.1, we obtain

$$\phi_1^+ + q\,\phi_2^+ = \tilde{r}(\phi_1^- + q\,\phi_2^-). \tag{5.5}$$

Taking into account that $q\,\phi_2^- \in L_2^-$ in this case, we see that

$$\phi_1^+ + q\,\phi_2^+ = \frac{K}{\xi + b}, \tag{5.6}$$

$$\phi_1^- + q\,\phi_2^- = \frac{K}{\xi - b}, \tag{5.7}$$

where $K \neq 0$ due to the condition $\phi_1^-(-b) \neq 0$. In fact, if $K = 0$, it follows from (5.6) and the first equation in (5.4) that $\phi_2^+ = \phi_1^- = 0$. $\square$

Since $(\xi + b)\phi^+$ and $(\xi - b)\phi^-$, with ϕ^+, ϕ^- satisfying the assumptions of Theorem 5.1, can be taken as one of the columns (the first, for instance) in G_+^{-1} and G_-, respectively, this result will allow us to obtain the other column.

It is easy to see that the left-hand sides of the two equalities in (5.3) represent the determinants of

$$M_+ = \begin{bmatrix} (\xi + b)\phi_1^+ & -q \\ (\xi + b)\phi_2^+ & 1 \end{bmatrix}, \quad M_- = \begin{bmatrix} (\xi - b)\phi_1^- & -q \\ (\xi - b)\phi_2^- & 1 \end{bmatrix}, \tag{5.8}$$

respectively. Moreover, defining

$$\tilde{\phi}_+ = \tilde{\phi}_- = \begin{bmatrix} -q \\ 1 \end{bmatrix}, \tag{5.9}$$

we see that (as it happens with the first column in M_+ and M_-) these satisfy an equality of the form $G\,\tilde{\phi}_+ = \tilde{\phi}_-$. Therefore we have $G\,M_+ = M_-$ and, since the results of Theorem 5.1 imply that $\det M_+$ does not vanish on $\overline{\mathbb{C}^+}$, we can write $G = M_-\,M_+^{-1}$. This is not a generalized factorization for G, due to the pole $\xi = -b$ in M_-, but we have the following result.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *be satisfied. A meromorphic factorization for G is*

$$G = M_-\,M_+^{-1}, \tag{5.10}$$

where $M_-,\ M_+$ are given by (5.8).

Proof. It follows from (5.8) and (5.3) that $r_+\,M_+^{\pm 1} \in \left(L_2^+\right)^{2\times 2}$. On the other hand, we see from (5.8) and (5.3) that $r_+^{-1}\,M_-^{\pm 1} \in \mathscr{H}\left(\mathbb{C}^-\right)$ and

$$r_-\left(q^{-1}\,M_-^{\pm 1}\right) \in \left(L_2^-\right)^{2\times 2}. \qquad \square$$

Now we can obtain a generalized factorization for G using the procedure presented in [2]. In fact, we have

$$M_- = \begin{bmatrix} (\xi - b)\,\phi_1^- & -q\,\tilde{r}^{-1} \\ (\xi - b)\,\phi_2^- & \tilde{r}^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{r} \end{bmatrix} = T_-\,R, \tag{5.11}$$

where T_- and R denote the first and the second factors on the middle side of (5.11), respectively.

Now $T_- \in \left(\mathscr{H}(\mathbb{C}^-)\right)^{2\times 2}$ but, since det $T_-(-b) = 0$, it follows that the two columns of $T_-(-b)$ are linearly dependent. Let t_1 and t_2 denote the first and the second columns in T_-, respectively. So there are complex constants λ_1 and λ_2 such that

$$\lambda_1\, t_1(-b) + \lambda_2\, t_2(-b) = 0.$$

Since

$$t_1(-b) = \begin{bmatrix} -2b\,\phi_1^-(-b) \\ 0 \end{bmatrix}, \quad t_2(-b) = \begin{bmatrix} -(b-a)/2b \\ 0 \end{bmatrix},$$

we can take

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{4b^2}{b-a}\,\phi_1^-\,(-b).$$

Let now

$$t_1' = t_1, \quad t_2' = \tilde{r}\,(\lambda_1\, t_1 + \lambda_2\, t_2)$$

and let T_-' be the matrix which is obtained from T_- when t_1 and t_2 are replaced by t_1' and t_2', respectively. We have $T_- = T_-'\, R_1$, where

$$T_-' = \begin{bmatrix} (\xi-b)\,\phi_1^- & (\xi-b)\tilde{r}\,\phi_1^- - \lambda_2\, q \\ (\xi-b)\,\phi_2^- & (\xi-b)\tilde{r}\,\phi_2^- + \lambda_2 \end{bmatrix}$$

satisfies the condition $r_-\left(T_-'\right)^{\pm 1} \in \left(L_2^-\right)^{2\times 2}$ and

$$R_1 = \begin{bmatrix} 1 & -\lambda_2^{-1} \\ 0 & \lambda_2^{-1}\,\tilde{r}^{-1} \end{bmatrix}.$$

Thus we have

$$G = M_-\, M_+^{-1} = T_-'(R_1\, R)M_+^{-1}$$

and since $R_+ = R_1\, R \in \mathscr{G}\left(L_\infty^+\right)^{2\times 2}$, we see that $G = G_-\, G_+$ is a canonical factorization for G with $G_-,\ G_+$ given by

$$G_- = T_-' = \begin{bmatrix} (\xi-b)\phi_1^- & (\xi-b)\tilde{r}\,\phi_1^- - \lambda_2\, q \\ (\xi-b)\phi_2^- & (\xi-b)\tilde{r}\,\phi_2^- + \lambda_2 \end{bmatrix}, \tag{5.12}$$

$$G_+^{-1} = M_+\, R_+^{-1} = \begin{bmatrix} (\xi+b)\phi_1^+ & (\xi-b)\phi_1^+ - \lambda_2\, q \\ (\xi+b)\phi_2^+ & (\xi-b)\phi_2^+ + \lambda_2 \end{bmatrix}. \tag{5.13}$$

Thus we have proved the following:

Theorem 5.3. *Let the assumptions of Theorem* 5.1 *be satisfied. A canonical factorization for* G *is* $G = G_-\, G_+$, *where* $G_-,\ G_+$ *are defined by* (5.12) *and* (5.13), *respectively*, *and*

$$\lambda_2 = -\frac{4b^2}{b-a}\,\phi_1^-(-b).$$

In order to obtain the factors $G_\pm$ we still have to determine a solution to $G\,\phi^+ = \tilde{r}\,\phi^-$ such that a condition of the form (5.2) is fulfilled.

This solution can be obtained from the results of Section 4, taking Remark 4.5 into account. In fact, solving (5.1) subject to condition (5.2) is equivalent to finding a solution to

$$G\,\phi^+ = \psi,$$

where $\phi^+ = \left(\phi_1^+,\ \phi_2^+\right) \in \left(L_2^+\right)^2,\ \psi = (\psi_1,\ \psi_2)$ with $\tilde{r}^{-1}\,\psi_1,\ \psi_2 \in L_2^-$. This merely corresponds to making $\tilde{r}\,\phi^- = \psi$.

It follows from Theorem 4.6 that $\phi_1^+,\ \phi_2^+,\ \psi_1$, and ψ_2 are given by (4.16), (4.17), (4.18) and (4.19), respectively.

Thus we have the following:

Theorem 5.4. *If $G \in \mathscr{N}_0$ is a matrix function of the form* (3.6) *such that condition* (4.2) *is satisfied and*

$$G\,\phi^+ = \tilde{r}\,\phi^-, \quad \phi^\pm \in (L_2^\pm)^2, \quad \phi_1^-(-b) \neq 0, \quad \phi_2^-(-b) = 0,$$

then ϕ_1^+, ϕ_2^+ are given by (4.16) *and* (4.17), *respectively*, *and*

$$\phi_1^- = K\,\frac{1 + (\xi + a)\,f^-}{\xi - b}, \quad \phi_2^- = -K\,\tilde{r}^{-1}\,f^- \tag{5.14}$$

with $f^- = P^-\,(\gamma/(\xi + a))$.

Corollary 5.5. *With the same assumptions of Theorem* 5.4, *a canonical factorization for G is $G = G_-\,G_+$, where*

$$G_- = \begin{bmatrix} 1 + (\xi + a)f^- & \tilde{r}\left(1 + (\xi + a)f^-\right) - \lambda_2\,q \\ -(\xi + b)\,f^- & -(\xi - b)f^- + \lambda_2 \end{bmatrix},$$

$$G_+^{-1} = \begin{bmatrix} 1 - (\xi + a)f^+ & \tilde{r}\left(1 - (\xi + a)f^+\right) - \lambda_2\,q \\ (\xi + b)f^+ & (\xi - b)f^+ + \lambda_2 \end{bmatrix}$$

with

$$f^\pm = P^\pm\left(\frac{\gamma}{\xi + a}\right) \quad \text{and} \quad \lambda_2 = 2b\left(\frac{1}{b-a} - f^-(-b)\right).$$

An interesting question remains to be studied, which concerns the form of the factors G_- and G_+.

It is clear that, in general, the factors given in Corollary 5.5 do not belong to the same group $\mathscr{N}_0$ to which the matrix G belongs. However, other canonical factorizations for G, in which the factors differ from those presented in the last theorem by constant multiplicative factors, are possible:

$$G = G_- \, G_+ = (G_- \, A) \, (A^{-1} \, G_+) = \tilde{G}_- \, \tilde{G}_+ \tag{5.15}$$

$\big(A \in \mathscr{G}(\mathbb{C}^{2\times 2})\big)$. The factors commute when they belong to $\mathscr{N}_0$.

In order to determine necessary and sufficient conditions for existence of such a commutative factorization, we look at the structure of matrix functions belonging to $\mathscr{N}_0$.

It is easy to see that, if $G \in \mathscr{N}_0$, the second column of G can be obtained from the first by rational transformation

$$\begin{bmatrix} q\,\gamma \\ 1-\gamma \end{bmatrix} = \begin{bmatrix} 0 & -q^2 \\ 1 & 2\,q \end{bmatrix} \begin{bmatrix} 1+\gamma \\ -q^{-1}\,\gamma \end{bmatrix}. \tag{5.16}$$

Let $G = G_- \, G_+$ be a canonical factorization in which $G_\pm \in \mathscr{N}_0$ and let $G_- = [g_{ij}^-]$, $G_+^{-1} = [g_{ij}^+]$.

Let moreover

$$\phi^\pm = \begin{bmatrix} \phi_1^\pm \\ \phi_2^\pm \end{bmatrix} = \frac{1}{\xi \pm b} \begin{bmatrix} g_{11}^\pm \\ g_{21}^\pm \end{bmatrix}, \quad \psi^\pm = \begin{bmatrix} \psi_1^\pm \\ \psi_2^\pm \end{bmatrix} = \frac{1}{\xi \pm b} \begin{bmatrix} g_{12}^\pm \\ g_{22}^\pm \end{bmatrix}. \tag{5.17}$$

We have

$$G\,\phi^+ = \tilde{r}\,\phi^-, \quad G\,\psi^+ = \tilde{r}\,\psi^-. \tag{5.18}$$

Since, according to (5.16), $\phi^\pm$ and $\psi^\pm$ must be related by

$$\begin{bmatrix} \psi_1^\pm \\ \psi_2^\pm \end{bmatrix} = \begin{bmatrix} 0 & -q^2 \\ 1 & 2\,q \end{bmatrix} \begin{bmatrix} \phi_1^\pm \\ \phi_2^\pm \end{bmatrix} \tag{5.19}$$

we see that ϕ_2^- must have a double zero for $\xi = -b \in \mathbb{C}^-$, to compensate the double pole of q^2. This means, in particular, that we must have $\phi_2^-(-b) = 0$, $\phi_1^-(-b) \neq 0$ and therefore it follows from Theorem 5.4 that ϕ_2^- is given by (5.14). Thus, the condition of existence of a double zero of ϕ_2^- for $\xi = -b$ can be expressed in the form

$$f^-(-b) = 0. \tag{5.20}$$

Conversely, if (5.20) is satisfied, condition (4.2) is also satisfied and G admits a canonical generalized factorization. Moreover, defining

$$G_- = (\xi - b) \begin{bmatrix} \phi_1^- & \psi_1^- \\ \phi_2^- & \psi_2^- \end{bmatrix}, \quad G_+^{-1} = (\xi + b) \begin{bmatrix} \phi_1^+ & \psi_1^+ \\ \phi_2^+ & \psi_2^+ \end{bmatrix},$$

where $\phi_1^\pm$, $\phi_2^\pm$ are defined as in Theorem 5.4 and $\psi_1^\pm$, $\psi_2^\pm$ are given by (5.19), we see that $G = G_- \, G_+$ is a canonical factorization for G, whose factors belong to $\mathscr{N}_0$.

We have thus proved the following:

Theorem 5.6. *Condition* (5.20) *is necessary and sufficient for existence of a canonical factorization for* G *in the group* $\mathscr{N}_0$. *If this condition is satisfied*, *we have* $G = G_- \, G_+$, *where*

$$G_- = \begin{bmatrix} 1+(\xi+a)f^- & q(\xi+a)f^- \\ -(\xi+b)f^- & 1-(\xi+a)f^- \end{bmatrix},$$
$$G_+^{-1} = \begin{bmatrix} 1-(\xi+a)f^+ & -q(\xi+a)f^+ \\ (\xi+b)f^+ & 1+(\xi+a)f^+ \end{bmatrix},$$

$f^{\pm}$ *being defined in Corollary* 5.5.

6. Non-canonical factorization

Let now $G \in \mathscr{N}_0$ admit a non-canonical factorization, which means that

$$\left[P^-\left(\frac{\gamma}{\xi+a}\right)\right]_{(-b)} = \frac{1}{b-a}. \tag{6.1}$$

According to Corollary 4.3, such a factorization for G is of the form

$$G = G_- \begin{bmatrix} \tilde{r}^{-1} & 0 \\ 0 & \tilde{r} \end{bmatrix} G_+, \tag{6.2}$$

where $\tilde{r}$ is given by $\tilde{r}(\xi) = (\xi-b)/(\xi+b)$ with $b \in \mathbb{C}^+$ (see Section 5).

To determine the factors $G_\pm$ we can use once again the solutions of Eq. (4.1).

If (ϕ^+, ϕ^-) is a non-trivial solution to $G\phi^+ = \phi^-$ and since $\dim\ker T_G = 1$, we can take $(\xi+b)\phi^+$ (resp. $(\xi-b)\phi^-$) as the first column in G_+^{-1} (resp. G_-).

We remark that, in the case we are considering, this result can be found in [7], in a different context: to determine a basis of $\ker T_G$ from a generalized factorization of G. Here we adopt the reverse perspective, since we know a basis of $\ker T_G$ and our purpose is to determine one of the columns in the factors $G_\pm$ of a generalized factorization for G.

In fact, we have in this case

$$\operatorname{diag}(1,\tilde{r})\, G_+\phi^+ = \operatorname{diag}(\tilde{r},1)\, G_-^{-1}\phi^- \tag{6.3}$$

and defining $G_+\phi^+ = \psi^+ = (\psi_1^+, \psi_2^+)$, $G_-^{-1}\phi^- = \psi^- = (\psi_1^-, \psi_2^-)$ it follows that

$$\psi_1^+ = \tilde{r}\,\psi_1^- = \frac{K_0}{\xi+b},$$
$$\tilde{r}\,\psi_2^+ = \psi_2^- = 0,$$

where $K_0 \in \mathbb{C}\backslash\{0\}$. Thus

$$G_+\phi^+ = \frac{K_0}{\xi+b}\begin{bmatrix}1\\0\end{bmatrix}, \quad G_-^{-1}\phi^- = \frac{K_0}{\xi-b}\begin{bmatrix}1\\0\end{bmatrix},$$

and we have

$$(\xi + b)\phi^+ = K_0\, G_+^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (\xi - b)\phi^- = K_0\, G_- \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where we can choose $K_0 = 1$.

On the other hand, it was shown in the proof of Theorem 4.1 that these first columns of G_+^{-1} and G_- satisfy the following relations:

$$\phi_1^+ + q\,\phi_2^+ = \phi_1^- + q\,\phi_2^- = \frac{K}{\xi + b} \tag{6.4}$$

with $K \in \mathbb{C}\backslash\{0\}$. This allows us to determine the second column in G_+^{-1} and G_-, as stated in the following theorem.

Theorem 6.1. *Let $G \in \mathscr{N}_0$ be such that* (6.1) *is satisfied. A non-canonical generalized factorization for G is $G = G_-\, \mathrm{diag}\,(\tilde{r}^{-1}, \tilde{r})\, G_+$ with*

$$G_- = \begin{bmatrix} \tilde{r}\left[1 + (\xi + a)\, f^-\right] & -(\xi + a)/(\xi - b) \\ (\xi - b)\, f^- & \tilde{r}^{-1} \end{bmatrix}, \tag{6.5}$$

$$G_+^{-1} = \begin{bmatrix} 1 - (\xi + a)\, f^+ & -q \\ (\xi + b)\, f^+ & 1 \end{bmatrix}, \tag{6.6}$$

where $f^\pm = P^\pm(\gamma/(\xi + a))$.

Proof. Let (ϕ^+, ϕ^-) be a non-trivial solution to the equation $G\,\phi^+ = \phi^-$, given by (4.12)–(4.15), with $K \neq 0$. We have

$$(\xi + b)\,(\phi_1^+ + q\,\phi_2^+) = K,$$
$$(\xi - b)\left(\tilde{r}^{-1}\,\phi_1^- + \frac{\xi + a}{\xi - b}\,\phi_2^-\right) = K.$$

This can be expressed in the form

$$\det \begin{bmatrix} (\xi + b)\,\phi_1^+ & -q \\ (\xi + b)\,\phi_2^+ & 1 \end{bmatrix} = K, \tag{6.7}$$

$$\det \begin{bmatrix} (\xi - b)\,\phi_1^- & -(\xi + a)/(\xi - b) \\ (\xi - b)\,\phi_2^- & \tilde{r}^{-1} \end{bmatrix} = K. \tag{6.8}$$

Let $\tilde{G}_+$ and $\tilde{G}_-$ denote the matrix functions on the left-hand sides of (6.7) and (6.8), respectively.

Since (ϕ^+, ϕ^-), with $\phi^\pm = (\phi_1^\pm, \phi_2^\pm)$, is a non-trivial solution to the equation $G\,\phi^+ = \phi^-$ and

$$G\,\begin{bmatrix} -q & 1 \end{bmatrix}^{\mathrm{T}} = \tilde{r}\,\begin{bmatrix} -\dfrac{\xi + a}{\xi - b} & \tilde{r}^{-1} \end{bmatrix}^{\mathrm{T}}$$

we see that

$$G\,\tilde{G}_+ = \tilde{G}_- \begin{bmatrix} \tilde{r}^{-1} & 0 \\ 0 & \tilde{r} \end{bmatrix}.$$

Moreover, it follows from (6.7) and (6.8) that $\det \tilde{G}_\pm = K \neq 0$. Therefore a non-canonical factorization for G is $G = G_- \operatorname{diag}(\tilde{r}^{-1}, \tilde{r}) G_+$, with $G_- = \tilde{G}_-$ and $G_+ = \tilde{G}_+^{-1}$. $\square$

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